M-math 2nd year Final Exam
Subject : Fourier Analysis

Time : 3.00 hours Max.Marks 60.

1. Let $\psi \in L^2(\mathbb{R}) \bigcap L^1(\mathbb{R})$ be a continuum wavelet i.e it satisfies $\int_{\mathbb{R}} |\hat{\psi}(x)|^2 \frac{dx}{|x|} < \infty$, where $\hat{\psi}$ is the Fourier transform of ψ. Show that $\int_{\mathbb{R}} \psi(x)\, dx = 0$. (10)

2. Let $d > 1$ be an integer. Show that the trigonometric polynomials are dense in $L^1(T^d)$, where $T^d := (0,1)^d$, the d-fold Cartesian product of $(0,1)$. You may assume the result for $d = 1$. (10)

3. Let f be locally integrable with respect to Lebesgue measure λ on $\mathbb{R}^d$. Define

$$Mf(x) := \sup_{x \in B} \frac{1}{\lambda(B)} \int_B f\, d\lambda$$

where the supremum is over all balls B containing x. Show that $Mf(x)$ is a Borel measurable function on $\mathbb{R}^d$. (10)

4. a) Let $2k > d$ where k and d are positive integers. Let $\Delta := \sum_{i=1}^{d} \partial_i^2$ be the Laplacian. Show that if $\partial^\alpha f$ is continuous in $T^d := [0,1]^d$ and has period one for every multi index $\alpha, |\alpha| \leq 2k$, then the Fourier series for f viz. $\sum_{n \in Z^d} \hat{f}(n) e^{-2\pi i n \cdot x}$ converges uniformly in T^d.
Hint : Compute the Fourier transform $\int_{T^d} (I - \Delta)^k f(x) e^{-2\pi i n \cdot x} dx$ in terms of $\hat{f}(n)$. (10)
b) In a) does the Fourier series for f converge to f? Justify your answer. (5)

5. Let $\mathcal{S}$ be the Schwartz space on $\mathbb{R}$ and for $f \in \mathcal{S}$, let Hf denote its Hilbert transform. Show that for each $x \in \mathbb{R}$, the limit $\lim_{\epsilon \to 0, M \to \infty} \frac{1}{\pi} \int_{\epsilon < |y| < M} \frac{f(x-y)}{y} dy$ exists and for a.e. x, $Hf(x) = \lim_{\epsilon \to 0, M \to \infty} \frac{1}{\pi} \int_{\epsilon < |y| < M} \frac{f(x-y)}{y} dy$. (15)